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GENERALIZED POTENTIALS IN A DIFFERENTIAL GAME WITH A FIXED TERMINATION INSTANT

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A position differential game with a fixed termination instant is examined, the payoff in which is the value of a specified function of the final state. A family of functions which are treated, in accord with the dynamic programming method, as generalized potentials is defined. It is shown that a function of the differential game's value coincides the lower envelope of this family of generalized potentials. The problem on the existence of players' strategies optimal in-the-large is investigated as well. The material in this paper adjoins the investigations in [1 — 9].

1. Let a controlled system's motion be described by the differential equation

$$x^{\cdot} = f(t, x, u, v), \quad f: [t_0, \vartheta] \times R^n \times P \times Q \rightarrow R^n \tag{1.1}$$

$$t \in [t_0, \vartheta], \quad x \in R^n, \quad u \in P \subset R^p, \quad v \in Q \subset R^q$$

Here P and Q are compacts; f is a continuous function satisfying a Lipschitz condition in variable x in each domain $[t_0, \vartheta] \times D \times P \times Q$, where D is some bounded set in R^n. It is assumed that the initial points $x_0 = x[t_0]$ belong to some compactum X_{t_0}. The symbol X_τ denotes the aggregate of all points x_τ for which a solution exists for the contingency equation

$$x^{\cdot}[t] \in \mathrm{co}\,\{f(t, x, u, v): u \in P, \ v \in Q\}, \quad t_0 \leqslant t \leqslant \tau$$

$$x[t_0] \in X_{t_0}, \quad x[\tau] = x_\tau$$

It is assumed that the sets X_τ $(t_0 \leqslant \tau \leqslant \vartheta)$ are nonempty and uniformly bounded, we shall use the following notation:

$$H = \{(t, x): t_0 \leqslant t \leqslant \vartheta, \quad x \in X_t\}$$

We examine a differential game in which the payoff is the quantity $\sigma(x[\vartheta])$, where σ: $R^n \rightarrow R$ is a specified continuous function and $x[\vartheta]$, is the system's phase state realized at the final instant $t = \vartheta$.

It is assumed that the first player, who has control u at his disposal, selects pure position strategies $U \div u(t, x)$ and strives to minimize the payoff's value and the second player, who has control v at his disposal, selects the counterstrategies $V^u \div v(t, x, u)$ and strives to maximize $\sigma(x[\vartheta])$. In this game there exists, for each initial position $(t_*, x_*) \in H$, a saddle point which is formed by the pair (U_e, V_e^u), where U_e and V_e^u are a pure strategy and counterstrategy, extremal to the bridges

$$W_{u_*}^{\circ} = \{(t, x) \in H: c_0(t, x) \leqslant c_0(t_*, x_*)\} \tag{1.2}$$

$$W_v^\circ = \{(t, x) \in H: c_0(t, x) \geqslant c_0(t_*, x_*)\} \tag{1.3}$$

respectively [3]. Here $c_0: H \to R$ is a function of the game's value. It is well known also that this function can be constructed in various ways [3 − 9]. In the present paper we show that function c_0 can be determined as the lower envelope of a certain family of generalized potentials.

2. Let us define the family of generalized potentials. For the point (t^*, t_*, x_*, u_*), where $(t_*, x_*, u_*) \in H \times P$ and $t^* \in [t, \vartheta]$, the symbol $G(t^*, t_*, x_*, u_*)$ denotes the aggregate of points x^* for which a solution exists of the contingency equation

$$x^{\cdot}[t] \in c_0\{f(t, x[t], u_*, v): v \in Q\}, \quad t_* \leqslant t \leqslant t^*$$
$$x[t_*] = x_*, \quad x[t^*] = x^*$$

(i.e., $G(t^*, t_*, x_*, u_*)$ is the closure of the attainability domain for system (1.1) under a constant control $u(t) = u_*$ and all possible program controls $v(t) \in Q$, $t_* \leqslant t \leqslant t^*$).

A function $\omega: H \to R$ is called a generalized potential if the following three conditions are fulfilled for it:

1°. Function ω is continuous on H.

2°. The boundary condition

$$\omega(\vartheta, x) = \sigma(x) \quad \text{for } x \in X \tag{2.1}$$

is satisfied.

3°. The inequality

$$\lim_{t^* \to t_* + 0} \min_{u \in P} \max_{x \in G(t^*, t_*, x_*, u)} \frac{\omega(t^*, x) - \omega(t_*, x_*)}{t^* - t_*} \leqslant 0 \tag{2.2}$$

is valid for any point $(t_*, x_*) \in H$, $t_* < \vartheta$.

The aggregate of functions ω satisfying these three conditions is denoted Ω. We indicate certain statements valid for set Ω.

Lemma 2.1. Set Ω is nonempty.

To prove this it suffices to consider the function

$$\omega(t_*, x_*) = \min_{u \in P} \max_{x \in G(\vartheta, t_*, x_*, u)} \sigma(x), \quad (t_*, x_*) \in H \tag{2.3}$$

and to verify directly the fulfillment of conditions 1° − 3° for it.

Lemma 2.2. For any function $\omega_* \in \Omega$ and any number $\tau \in [t_0, \vartheta]$ the function ω^* defined by the relation

$$\omega^*(t_*, x_*) = \begin{cases} \omega_*(t_*, x_*) & \text{for } t_* \in [\tau, \vartheta], \ x_* \in X_{t_*} \\ \min\limits_{u \in P} \max\limits_{x \in G(\tau, t_*, x_*, u)} \omega_*(\tau, x) & \\ & \text{for } t_* \in [t_0, \tau], \ x_* \in X_{t_*} \end{cases} \tag{2.4}$$

belongs to set Ω.

Lemma 2.3. The lower envelope ω_* of any finite collection of functions $\omega_i \in \Omega$ $(i = 1, 2, \ldots, m)$ defined by the equality

$$\omega_*(t_*, x_*) = \min_{1 \leqslant i \leqslant m} \omega_i(t_*, x_*), \ (t_*, x_*) \in H \tag{2.5}$$

belongs to set Ω.

Lemma 2.2 and 2.3 are proved by simple verification of conditions $1° - 3°$ for the functions ω^* of (2.4) and ω_* of (2.5).

For the next statement we introduce some notation. Let $x \in X_\vartheta$ and $\beta > 0$; we set

$$d_*(x, \beta) = \sigma(x) - \min\{\sigma(y)\}: y \in X_\vartheta \cap S(x, \beta)\}$$
$$S(x, \beta) = \{y \in R^n : \| y - x \| \leqslant \beta\}$$
$$d^*(\beta) = \max\{d_*(x, \beta) : x \in X_\vartheta\}$$
$$d(\alpha) = d^*(\alpha \exp \lambda(\vartheta - t_0)) \quad (\alpha > 0)$$

where λ is the Lipschitz constant with respect of variable x in the domain $H \times P \times Q$ for function f. We note that the continuity of function σ implies $d(\alpha) \to 0$ as $\alpha \to 0$.

Lemma 2.4. For any points $(t_*, x_*) \in H$ and $(t_*, x^*) \in H$ and for every function $\omega \in \Omega$ we can construct a function $\omega° \in \Omega$ satisfying the inequality

$$\omega°(t_*, x^*) \leqslant \omega(t_*, x_*) + d(\| x_* - x^* \|) \tag{2.6}$$

We present the proof of this lemma. Let

$$r(t) = \| x_* - x^* \| \exp \lambda(t - t_*)$$

We consider the function $\eta : H \to R$ defined by the equality

$$\eta(t, x) = \min\{\omega(t, y) : y \in X_t \cap S(x, r(t))\} + d(\| x_* - x^* \|)$$

It can be proved that function η satisfies the inequalities

$$\eta(t_*, x^*) \leqslant \omega(t_*, x_*) + d(\| x_* - x^* \|), \quad \eta(\vartheta, x) \geqslant \sigma(x) \quad \text{for } x \in X_\vartheta$$

is continuous and that condition 3° is satisfied for it. Further, it can be verified that the function

$$\omega°(t, x) = \min\{\eta(t, x), \omega(t, x)\}, \quad (t, x) \in H$$

belongs to class Ω and satisfies inequality (2.6).

We investigate the lower envelope of family Ω

$$\omega_0(t, x) = \inf\{\omega(t, x) : \omega \in \Omega\}, \quad (t, x) \in H \tag{2.7}$$

The following statement is valid.

Theorem 2.1. The function ω_0 of (2.7) is upper-semicontinuous on H; the function ω_0 is continuous in x on X_t for each $t \in [t_0, \vartheta]$; the relations

$$\omega_0(\vartheta, x) = \sigma(x), \quad x \in X_\vartheta \tag{2.8}$$

$$\inf_{t^* \in [t^*, \vartheta]} \min_{u \in P} \max_{x \in G(t^*, t_*, x_*, u)} \omega_0(t^*, x) \geqslant \omega_0(t_*, x_*) \tag{2.9}$$

are valid. $\quad \forall (t_*, x_*) \in H, \quad t_* < \vartheta$

Proof. The upper-semicontinuity of function ω_0 follows directly from its definition as the lower envelope of the set Ω of continuous functions. The continuity in x of function ω_0 derives from Lemma 2.4. Equality (2.8) follows immediately from (2.1) and (2.7). Let us prove inequality (2.9). We assume the contrary. Let a point $(t_*, x_*) \in H$, the numbers $\tau \in (t_*, \vartheta]$ and $\alpha > 0$ and the control $u_* \in P$ exist such that

$$\max_{x \in G(\tau, t_*, x_*, u_*)} \omega_0(\tau, x) \leqslant \omega_0(t_*, x_*) - 3\alpha \tag{2.10}$$

By the definition of function ω_0 we can find, for any point $x^* \in G^* = G(\tau, t_*, x_*, u_*)$, a function $\omega(\cdot \mid x^*) \in \Omega$ so as to fulfill the inequality $\omega(\tau, x^* \mid x^*) \leqslant \omega_0(\tau, x^*) + \alpha$. The functions $\omega_0(\tau, x)$ and $\omega(\tau, x \mid x^*)$ are continuous in x; therefore, a number $\beta(x^*) > 0$ exists for any point $x^* \in G^*$, such that

$$\omega(\tau, x \mid x^*) \leqslant \omega_0(\tau, x) + 2\alpha \quad \text{for} \quad x \in S(x^*, \beta(x^*))$$

We obtain a covering of compactum G^* by spheres $S(x^*, \beta(x^*))$. From this covering we can separate a finite subcovering S_i $(i = 1, \ldots, m)$ so that

$$\omega_i(\tau, x) \leqslant \omega_0(\tau, x) + 2\alpha, \quad x \in S_i \quad (i = 1, 2, \ldots, m) \tag{2.11}$$

where the ω_i are some functions from Ω. Let ω_* be the lower envelope of the aggregate $\{\omega_i, i = 1, 2, \ldots, m\}$. According to Lemma 2.3, $\omega_* \in \Omega$, and inequality

$$\omega_*(\tau, x) \leqslant \omega_0(\tau, x) + 2\alpha, \quad x \in G^* \tag{2.12}$$

follows from (2.11). We introduce into consideration the function ω^* of (2.4) for the function ω_*. According to Lemma 2.2, $\omega^* \in \Omega$. The inequality $\omega^*(t_*, x_*) \leqslant \max\{\omega_*(\tau, x^*) : x^* \in G^*\}$ follows directly from the definition of set $G^* = G(\tau, t_*, x_*, u_*)$ and of function ω^* of (2.4). Allowing for estimates (2.12) and (2.10), this inequality can be prolonged as follows:

$$\omega^*(t_*, x_*) \leqslant \max_{x^* \in G^*} \omega_*(\tau, x^*) \leqslant \max_{x^* \in G^*} \omega_0(\tau, x^*) + 2\alpha \leqslant \omega_0(t_*, x_*) - \alpha$$

Thus, we find that a function $\omega^* \in \Omega$ exists for which $\omega^*(t_*, x_*) < \omega_0(t_*, x_*)$. We have arrived at a contradiction with the definition of function ω_0. Inequality (2.9) and Theorem 2.1 have been proved.

3. Let us show that

$$\omega_0(t_*, x_*) = c_0(t_*, x_*), \quad (t_*, x_*) \in H \tag{3.1}$$

where $c_0(t_*, x_*)$ is the differential game's value in the class of pure position strategies $U \div u(t, x)$ and counterstrategies $V^u \div v(t, x, u)$ for the initial position (t_*, x_*). At first we present the following statement.

Lemma 3.1. The set

$$W = \{(t, x) \in H: \omega(t, x) \leqslant c\} \tag{3.2}$$

is u_*-stable [3] for any number c and any function $\omega \in \Omega$.

To prove this lemma we can consider the set

$$W_\alpha = \{(t, x) \in H: \omega(t, x) \leqslant c + \alpha(t - t_0)\} \quad (\alpha > 0)$$

and, using (2.2), verify that it is u_*-stable. Then the u_*-stability of the set W of (3.2) can be obtained as a consequence of the u_*-stability of set W_α in the limit as $\alpha \to 0$.

Also valid is the following

Lemma 3.2. The set

$$W^{0,c} = \{(t, x) \in H: \omega_0(t, x) \leqslant c\} \tag{3.3}$$

is v-stable [3] for any number c.

The validity of this lemma follows immediately from (2.9). We make use of Lemmas 3.1 and 3.2 to prove equality (3.1). For a specified position (t_*, x_*) and for some number $\beta > 0$ we define a function ω_β so as to fulfill the inequality $\omega_\beta(t_*, x_*) \leqslant \omega_0(t_*, x_*) + \beta$. We construct a position strategy $U_\beta \div u_\beta(t, x)$ extremal to set W of (3.2) wherein $\omega = \omega_\beta$ and $c = \omega_\beta(t_*, x_*)$. Then, according to the results in [3], the inequality

$$\max\{\sigma(x): x \in X[\vartheta; t_*, x_*, U_\beta]\} \leqslant \omega_0(t_*, x_*) + \beta \tag{3.4}$$

is valid for this strategy.

Here and below $X[\vartheta; t_*, x_*, U]$ and $X[\vartheta, t_*, x_*, V^u]$ are the sets of points $x[\vartheta; t_*, x_*, U]$ and $x[\vartheta; t_*, x_*, V^u]$ that are realized at instant ϑ by all possible motions generated by strategies $U \div u(t, x)$ and counterstrategies $V^u \div v(t, x, u)$, respectively. On the other hand, the inequality

$$\min\{\sigma(x) : x \in X[\vartheta; t_*, x_*, V_e^u]\} \geqslant \omega_0(t_*, x_*) \tag{3.5}$$

is valid for the counterstrategy $V_e^u \div v(t, x, u)$ extremal to the set $W^{0,c}$ of (3.3) wherein $c = \omega_0(t_*, x_*)$ As $\beta \to 0$, from (3.4) and (3.5) we obtain

$$\inf_U \max\{\sigma(x): x \in X[\vartheta; t_*, x_*, U]\} = \max_{V^u} \min\{\sigma(x): x \in X[\vartheta; t_*, x_* V^u]\} \tag{3.6}$$

Hence it follows that the quantity $\omega_0(t_*, x_*)$ coincides with the value $c_0(t_*, x_*)$ of the differential game in the class of strategies $U \div u(t, x)$ and counterstrategies $V^u \div v(t, x, u)$.

It is well known that not only all sets W of (3.2) but also the set

$$W_c^\circ = \{(t, x) \in H: \omega_0(t, x) = c_0(t, x) \leqslant c\} \tag{3.7}$$

corresponding to the lower envelope $\omega_0 = c_0$ and to any number c are u_*-stable. The lower bound in (3.6) is reached by the strategy $U_e \div u_e(t, x)$ extremal to the set W_c° of (3.7) with $c = c_0(t_*, x_*)$. It is well known also that the function $c_0 = \omega_0$ is continuous on H.

We note the following circumstance. As shown above, the u_*-stability of the set W of (3.2) follows from the condition $\omega \in \Omega$. However, the converse is false, i.e., the membership of a function ω_0 to class Ω does not follow from the u_*-stability of set W_c° of (3.7). As an example, where the lower envelope ω_0 does not satisfy condition 3°, we can cite the well-known game [1] specified by the equation

$$x_1^\cdot = x_2 + v,\ x_2^\cdot = u,\ |u| \leqslant 1,\ |v| \leqslant 1,\ 0 = t_0 \leqslant t \leqslant \vartheta = 2$$

and by the function $\sigma(x) = |x_1|$. Here

$$\lim_{t^* \to t_* + 0} \min_{u \in P} \max_{x \in G(t^*, t_*, x_*, u)} \frac{\omega_0(t^*, x) - \omega_0(t_*, x_*)}{t^* - t_*} = 2 - t_* > 0$$

at points (t_*, x_*), where $t_* \in [1, 2)$ and $x_{*1} + (2 - t)x_{*2} = 0$.

4. Let us consider the case when $c_0 = \omega_0 \in \Omega$. In this case we construct a position strategy $U^\circ \div u^\circ(t, x)$ optimal in-the-large, for which the inequality $\sigma(x[\vartheta]) \leqslant c_0(\tau, x[\tau])$ is valid for any motion $x[t] = x[t; t_*, x_*, U^\circ]$ and any instant $\tau \in [t_*, \vartheta]$. We note that the strategy U_e extremal to the set $W_{u_*}^\circ$ of (1.2) ensures the fulfillment of the inequality $\sigma(x[\vartheta]) \leqslant c_0(t_*, x_*)$ for the specified initial position (t_*, x_*); but an instant $\tau \in [t_*, \vartheta]$ and a motion $x[t] = x[t; t_*, x_*, U_e]$ can exist such that $\sigma(x[\vartheta]) > c_0(\tau, x[\tau])$.

It is well known that the strategy $U^\circ \div u^\circ(t, x)$, optimal in-the-large, can be constructed when the function c_0 is continuously differentiable in t and in x at points $(t, x) \in H$ where $c_0(t, x) > \sigma_0 = \min\{\sigma(x) : x \in X_\vartheta\}$ [3]. As an example in which the function $c_0 = \omega_0$ is not continuously differentiable but does belong to the class Ω of generalized Bellman functions, we can cite the game specified by the equations $x_1^\cdot = x_3$, $x_2^\cdot = x_4$, $x_3^\cdot = u_1 - v_1$, $x_4^\cdot = u_2 - v_2$, $|u_1| \leqslant \lambda_1$, $|u_2| \leqslant \lambda_2$, $(v_1^2 + v_2^2)^{1/2} \leqslant \lambda_2$, $\lambda_1 < \lambda_2$ and by the function $\sigma(x) = x_1^2 + x_2^2$.

Thus, let $\omega_0 \in \Omega$. For a positive parameter α we define the functions $\delta_\alpha : H \to R$ and $u_\alpha : H \to P$ which associate with the point $(t_*, x_*) \in H$ a number $\delta_\alpha(t_*, x_*) > 0$ and a vector $u_\alpha(t_*, x_*) \in P$ satisfying the inequality

$$\max_x \omega_0(t_* + \delta_\alpha(t_*, x_*), x) \leqslant \omega_0(t_*, x_*) + \alpha\delta_\alpha(t_*, x_*)$$
$$x \in G(t_* + \delta_*(t_*, x_*), t_*, x_*, u_\alpha(t_*, x_*))$$

The existence of such functions δ_α and u_α follows from (2.2). We now assume that the function $u^\circ: H \to P$ associates a vector $u^\circ(t_*, x_*)$ with the point $(t_*, x_*) \in H$, which is the limit of some sequence $u_{\alpha_k}(t_*, x_*)$ $(k = 1, 2, \ldots)$, where $\alpha_k \to 0$ as $k \to \infty$ (the sequence of numbers α_k depends, in general, on the point (t_*, x_*)). To determine the motions generated by strategy $U^\circ \div u^\circ(t, x)$ we require the function $\delta_\varepsilon^\circ: H \to R$. This function is defined as follows: for a prescribed parameter $\varepsilon > 0$ and for the point $(t_*, x_*) \in H$ we determine a number $\alpha_* > 0$ so as to fulfil the inequality

$$\| u^\circ(t_*, x_*) - u_{\alpha_*}(t_*, x_*) \| < \varepsilon, \quad \alpha_* \leqslant \varepsilon$$

and we set $\delta_\varepsilon^\circ(t_*, x_*) = \delta_{\alpha_*}(t_*, x_*)$. We define the motion $x[t; t_*, x_*, U^\circ]$ $(t_* \leqslant t \leqslant \vartheta)$ generated by strategy $U^\circ \div u^\circ(t, x)$ as the limit of a sequence of Euler polygonal lines. However, in contrast to [3], wherein the partitionings Δ of the segment $[t_*, \vartheta]$ can be selected in advance at the initial instant, here we examine partitionings Δ that are formed during the game by the function $\delta_\varepsilon{}^\circ$. For a chosen parameter $\varepsilon > 0$ and for the control $v[t]$ $(t_* \leqslant t \leqslant \vartheta)$ realized by the second player, the approximate motion (Euler polygonal line) $x[t] = x[t; t_*, x_*, U^\circ, v[\cdot], \delta_\varepsilon{}^\circ]$ can be meaningfully determined as the motion generated by the second player's control $v[t]$ $(t_* \leqslant t \leqslant \vartheta)$ and by the first player's piecewise-constant control

$$u[t] = u^\circ(\tau', x[\tau']), \quad \tau' \leqslant t \leqslant \tau' + \delta_\varepsilon{}^\circ(\tau', x[\tau'])$$

Formally the approximate motion $x[t;\ t_*, x_*, U^\circ, v[\cdot], \delta_\varepsilon{}^\circ]$ is determined in accord with [5]. Then, every limit, uniform on $[t_*, \vartheta]$, of the sequence of approximate motions $x[t; t_*, x_*, U^\circ, v_k[\cdot], \delta_{\varepsilon_k}{}^\circ]$, where $\varepsilon_k > 0$ $(k = 1, 2, \ldots)$ is some sequence of numbers converging to zero, is called a motion $x[t;\ t_*, x_*, U^\circ]$. It can be shown that under the given definition of motions the strategy $U^\circ \div u^\circ(t, x)$ is optimal in-the-large.

When $c_0 = \omega_0 \notin \Omega$ we can construct a strategy $U^{(\varepsilon)} \div u^{(\varepsilon)}(t, x)$, ε-optimal in-the-large, for which the inequality $\sigma(x[\vartheta]) \leqslant c_0(t, x[t]) + \varepsilon$ is valid for any instant $t \in [t_*, \vartheta]$ and for every motion $x[t] = x[t; t_*, x_*, U^{(\varepsilon)}]$. The construction of this strategy and of the motions it generates can be given within the framework of an extremal construction [3].

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